Indian Statistical Institute, Bangalore

M. Math First Year

First Semester - Measure Theoretic Probability

Midterm Exam — Date: September 06, 2019

Maximum marks: 30 — Duration: 2 hours 15 minutes

Answer any three and each question carries 10 Marks

1. (i) Let X be an uncountable set and $\mathcal{A}$ be the collection of subsets A of X such that A or $X \setminus A$ is countable. Prove that $\mathcal{A}$ is a σ-algebra and smallest σ-algebra containing the singletons.

   (ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that $f^{-1}(B)$ is Lebesgue measurable for any Borel set B (*Marks: 5*).

2. (i) Prove that the outer measure of an interval is the length of the interval.

   (ii) Let $A = \mathbb{Q} \cap [0,1]$ and let $\{I_n\}$ be a finite collection of open intervals covering A. Prove that $\sum l(I_n) \geq 1$ (*Marks: 5*).

3. (i) Prove that the interval (a, ∞) is Lebesgue measurable (*Marks: 5*).

   (ii) Prove that sets of outer measure zero are Lebesgue measurable (*Marks: 2*).

   (iii) Prove that $m^*(A \cup B) = m^*(B)$ if $m^*(A) = 0$.

4. (i) Prove that a finitely additive and countably subadditive nonnegative real-valued function μ defined on a σ-algebra is a measure provided $\mu(\emptyset) = 0$.

   (ii) If $f \geq 0$ is a measurable function on a measure space $(X, \mathcal{A}, \mu)$ and $\int f = 0$, prove that $f = 0$ a.e. (*Marks: 5*).

5. (i) Let $(X, \mathcal{A}, \mu)$ be a complete finite measure space. If f is a bounded measurable function, then $\int f = \sup\{\int \phi \mid \phi \leq f$ and ϕ takes only finitely many values$\}$.

   (ii) Let $f \geq 0$ be a Lebesgue integrable function on $\mathbb{R}$. Prove that

$$m \times m(\{(x,y) \mid 0 \leq y \leq f(x)\}) = \int f(x)dm(x) = \int_0^\infty \phi(t)dm(t)$$

   where $\phi(t) = m(\{x \mid f(x) \geq t\}$ (*Marks: 5*).